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AN $\boldsymbol{E}$-BASED MIXED FORMULATION FOR A TIME-DEPENDENT EDDY CURRENT PROBLEM

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ABSTRACT. In this paper, we analyze a mixed form of a time-dependent eddy current problem formulated in terms of the electric field $\boldsymbol{E}$. We show that this formulation admits a well-posed saddle point structure when the constraints satisfied by the primary unknown in the dielectric material are handled by means of a Lagrange multiplier. We use Nédélec edge elements and standard nodal finite elements to define a semi-discrete Galerkin scheme. Furthermore, we introduce the corresponding backward-Euler fully-discrete formulation and prove error estimates.

1. Introduction

The numerical solution of Maxwell equations is now an increasingly important research area in science and engineering. We refer the reader to the books by Bossavit [9], Monk [19], and Silvester and Ferrari [23], as a representative sampling of text books devoted to the numerical solution of electromagnetic problems. Among the numerical methods found in the literature to approximate Maxwell equations, the finite element method is the most extended.

In applications related to electrical power engineering (see for instance [22]) the displacement current existing in a metallic conductor is negligible compared with the conduction current. In such situations the displacement currents can be dropped from Maxwell's equations to obtain a magneto-quasistatic submodel usually called the *eddy current problem*; see for instance [9, Chapter 8]. From the mathematical point of view, this submodel provides a reasonable approximation to the solution of the full Maxwell system in the low frequency range [3].

When dealing with alternating currents, the imposed current density shows a harmonic dependence on time. In such a case, the steady state electric and magnetic fields also have this harmonic behavior, leading to the so-called *time-harmonic* eddy current problem. However, even in the case of a sinusoidal supply voltage, on some occasions one may need to simulate transient states. Besides, in some cases it is

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not possible to assume a sinusoidal behavior for the whole electromagnetic system. Actually, the present paper is intended as a first (linear) step towards the nonlinear case that happens in the presence of ferromagnetic materials. In this approach, we allow the magnetic permeability to be time-dependent and write the problem in terms of the electric field $\boldsymbol{E}$. In contrast to the $\boldsymbol{H}$-based formulation given in [18], the $\boldsymbol{E}$-formulation fits well into the theory of monotone operators, because the reluctivity (the inverse of the magnetic permeability) appears as a diffusion coefficient in the degenerate parabolic problem at hand (see (3.14) below).

Generally, the eddy current problem is posed in the whole space with decay conditions on the fields at infinity. There exist many techniques to tackle this difficulty; for example, a BEM-FEM strategy is used in [15, 17] in the harmonic regime case and in [18] in the transient case. Here we opted for a simpler approach: we restrict the equations to a sufficiently large domain Ω containing the region of interest and impose a convenient artificial boundary condition on its border. Although thorough mathematical and numerical analyses of several finite element formulations of the time-harmonic eddy current model in a bounded domain have been performed (see for instance Bermúdez *et al.* [6] and Alonso Rodríguez *et al.* [1]), this is not the case for the time-dependent problem.

The aim of this work is to propose a new formulation for the time-dependent eddy current model posed in a bounded domain, with no restrictions on the topology of the conductor or on the regularity of its boundary. This formulation is obtained by introducing a time primitive of $\boldsymbol{E}$ as the primary unknown and using a Lagrange multiplier associated to the divergence-free constraint satisfied by this variable in the insulating region surrounding the conductor. The techniques used to show that this saddle-point formulation is well posed are similar to the ones given in [7, 18]. (Another formulation for a time-dependent eddy current problem in terms of a magnetic vector potential is given in [5].) Mixed finite element schemes have been used extensively for the approximation of evolution problems, mainly in fluid dynamics applications; see, for instance, Johnson and Thomée [16] and Bernardi and Raugel [7]. More recently, Boffi and Gastaldi [8] gave sufficient conditions for the convergence of approximation for two types of mixed parabolic problems, the heat equation in mixed form being a model for the first case, while the time-dependent Stokes problem fits into the second one.

We perform a space discretization of our weak formulation by using Nédélec edge elements (see [20]) for the main variable and standard finite elements for the Lagrange multiplier. We show that our semi-discrete Galerkin scheme is uniquely solvable and provide asymptotic error estimates in terms of the space discretization parameter h. We also propose a fully discrete Galerkin scheme based on a backward Euler time stepping. Here again we provide error estimates that prove optimal convergence. Moreover, we obtain error estimates for the eddy currents and the magnetic induction field.

The paper is organized as follows. In Section 2, we summarize some results from [10, 11, 12] concerning tangential traces in $\mathbf{H}(\mathbf{curl}, \Omega)$ and recall some basic results for the study of time-dependent problems. In Section 3, we introduce the model problem and show how to handle the constraint satisfied by the electric field in the insulator by means of a Lagrange multiplier. In Section 4, we prove that the resulting saddle point problem is uniquely solvable. The derivation of a semi-discretization in space and its convergence analysis are reported in Section 5.

Finally, a backward Euler method is employed to obtain a time discretization of the problem. The results presented in Section 6 prove that the resulting fully discrete scheme is convergent in an optimal way. We end this paper by summarizing its main results in Section 7.

2. Preliminaries

We use boldface letters to denote vectors as well as vector-valued functions, and the symbol $|\cdot|$ represents the standard Euclidean norm for vectors. In this section Ω is a generic Lipschitz bounded domain of $\mathbb{R}^3$. We denote by Γ its boundary and by $\boldsymbol{n}$ the unit normal outward to Ω. Let

$$(f,g)_{0,\Omega} := \int_\Omega fg$$

be the inner product in $\mathrm{L}^2(\Omega)$ and $\|\cdot\|_{0,\Omega}$ the corresponding norm. As usual, for all $s > 0$, $\|\cdot\|_{s,\Omega}$ stands for the norm of the Hilbertian Sobolev space $\mathrm{H}^s(\Omega)$ and $|\cdot|_{s,\Omega}$ for the corresponding seminorm. The space $\mathrm{H}^{1/2}(\Gamma)$ is defined by localization on the Lipschitz surface Γ. We denote by $\|\cdot\|_{1/2,\Gamma}$ the norm in $\mathrm{H}^{1/2}(\Gamma)$ and $\langle\cdot,\cdot\rangle_\Gamma$ stands for the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and its dual $\mathrm{H}^{-1/2}(\Gamma)$.

We denote by $\gamma : \mathrm{H}^1(\Omega)^3 \to \mathrm{H}^{1/2}(\Gamma)^3$ the standard trace operator acting on vectors and define the tangential trace $\gamma_\tau : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathrm{L}^2(\Gamma)^3$ as $\boldsymbol{q} \mapsto \gamma\boldsymbol{q} \times \boldsymbol{n}$. Extending the tangential trace by completeness to $\mathrm{H}^1(\Omega)^3$, we define the space $\mathbf{H}_\perp^{1/2}(\Gamma) := \gamma_\tau(\mathrm{H}^1(\Omega)^3)$ endowed with the norm

$$\|\boldsymbol{\lambda}\|_{\mathbf{H}_\perp^{1/2}(\Gamma)} := \inf_{\boldsymbol{q}\in\mathrm{H}^1(\Omega)^3} \left\{\|\boldsymbol{q}\|_{1,\Omega} :\ \gamma_\tau(\boldsymbol{q}) = \boldsymbol{\lambda}\right\},$$

which makes the linear mapping $\gamma_\tau : \mathrm{H}^1(\Omega)^3 \to \mathbf{H}_\perp^{1/2}(\Gamma)$ continuous and surjective. We refer to [10] for an intrinsic definition of $\mathbf{H}_\perp^{1/2}(\Gamma)$ in the case of a curvilinear Lipschitz polyhedron Ω. We now introduce the dual space $\mathbf{H}_\perp^{-1/2}(\Gamma)$ of $\mathbf{H}_\perp^{1/2}(\Gamma)$ with respect to the skew-symmetric pairing

$$\langle\boldsymbol{\lambda},\boldsymbol{\eta}\rangle_{\tau,\Gamma} := \int_\Gamma (\boldsymbol{\lambda}\times\boldsymbol{n})\cdot\boldsymbol{\eta} \qquad \forall\boldsymbol{\lambda},\boldsymbol{\eta} \in \mathrm{L}^2(\Gamma)^3.$$

The Green's formula

$$(2.1)\quad (\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} - (\mathbf{curl}\,\boldsymbol{u},\boldsymbol{v})_{0,\Omega} = \langle\gamma_\tau\boldsymbol{u},\gamma\boldsymbol{v}\rangle_{\tau,\Gamma} \quad \forall\boldsymbol{u}\in\mathcal{C}^\infty(\overline{\Omega})^3,\ \forall\boldsymbol{v}\in\mathrm{H}^1(\Omega)^3,$$

and the density of $\mathcal{C}^\infty(\overline{\Omega})^3$ in $\mathbf{H}(\mathbf{curl},\Omega)$ (see [19, Theorem 3.26]) prove that

$$\gamma_\tau :\ \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}_\perp^{-1/2}(\Gamma)$$

is continuous. A more accurate result is given by the following theorem.

Theorem 2.1. *Let*

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma) := \left\{\boldsymbol{\eta}\in\mathbf{H}_\perp^{-1/2}(\Gamma) :\ \mathrm{div}_\Gamma\,\boldsymbol{\eta}\in\mathrm{H}^{-1/2}(\Gamma)\right\}.$$

The operator $\gamma_\tau :\ \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)$ *is continuous, surjective, and has a continuous right inverse.*

Proof. See [11, Theorem 4.6] for the case of Lipschitz polyhedra (the proper definition of div_Γ can be found in the same reference, as well). The more general case of Lipschitz domains is shown in [12, Theorem 4.1]. □

The kernel of the tangential trace operator γ_τ in $\mathbf{H}(\mathbf{curl},\Omega)$ is the closed subspace

$$\mathbf{H}_0(\mathbf{curl},\Omega) := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl},\Omega) : \ \gamma_\tau \boldsymbol{v} = 0\}.$$

We will also use the normal trace $\gamma_{\boldsymbol{n}} : \mathcal{C}^\infty(\overline{\Omega})^3 \to \mathrm{L}^2(\Gamma)$ given by $\boldsymbol{q} \mapsto \gamma\boldsymbol{q}\cdot\boldsymbol{n}$. It is well known that this operator can be extended to a continuous and surjective mapping (cf. [19, Theorem 3.24])

$$\gamma_{\boldsymbol{n}} : \ \mathbf{H}(\mathrm{div},\Omega) \to \mathrm{H}^{-1/2}(\Gamma),$$

where $\mathbf{H}(\mathrm{div},\Omega) := \left\{\boldsymbol{q} \in \mathrm{L}^2(\Omega)^3 : \ \mathrm{div}\,\boldsymbol{q} \in \mathrm{L}^2(\Omega)\right\}$ is endowed with the graph norm.

Since we will deal with a time-domain problem, besides the Sobolev spaces defined above, we need to introduce spaces of functions defined on a bounded time interval $(0,T)$ and with values in a separable Hilbert space V, whose norm is denoted here by $\|\cdot\|_V$. We use the notation $\mathcal{C}^0([0,T];V)$ for the Banach space consisting of all continuous functions $f : \ [0,T] \to V$. More generally, for any $k \in \mathbb{N}$, $\mathcal{C}^k([0,T];V)$ denotes the subspace of $\mathcal{C}^0([0,T];V)$ of all functions f with (strong) derivatives $\dfrac{d^j f}{dt^j}$ in $\mathcal{C}^0([0,T];V)$ for all $1 \le j \le k$. In what follows, we will use indistinctly the notation

$$\frac{d}{dt} f = \partial_t f$$

to express the derivative with respect to the variable t.

We also consider the space $\mathrm{L}^2(0,T;V)$ of classes of functions $f : \ (0,T) \to V$ that are Böchner-measurable and such that

$$\|f\|^2_{\mathrm{L}^2(0,T;V)} := \int_0^T \|f(t)\|^2_V \, dt < +\infty.$$

Furthermore, we will use the space

$$\mathrm{H}^1(0,T;V) := \left\{ f : \ \exists g \in \mathrm{L}^2(0,T;V) \text{ and } \exists f_0 \in V \text{ such that} \right. \\ \left. f(t) = f_0 + \int_0^t g(s)\, ds \quad \forall t \in [0,T] \right\}.$$

Analogously, we define $\mathrm{H}^k(0,T;V)$ for all $k \in \mathbb{N}$.

3. Variational formulation

Our purpose is to determine the eddy currents induced in a three-dimensional conducting domain represented by the open and bounded set Ω_{c}, by a given time-dependent compactly-supported current density $\boldsymbol{J}$. We assume that Ω_{c} is a Lipschitz domain and denote by $\boldsymbol{n}$ the unit outward normal on $\Sigma := \partial\Omega_{\mathrm{c}}$. We denote by Σ_i, $i = 1, \ldots, I$, the connected components of Σ.

The electric and magnetic fields $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{H}(\boldsymbol{x},t)$ are solutions of a submodel of Maxwell's equations obtained by neglecting the displacement currents (see [3]):

$$\partial_t(\mu\boldsymbol{H}) + \mathbf{curl}\,\boldsymbol{E} = \mathbf{0} \quad \text{in } \mathbb{R}^3\times(0,T), \tag{3.1}$$

$$\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{J} + \sigma\boldsymbol{E} \quad \text{in } \mathbb{R}^3\times[0,T), \tag{3.2}$$

$$\operatorname{div}(\varepsilon\boldsymbol{E}) = 0 \quad \text{in } \left(\mathbb{R}^3\setminus\Omega_{\rm c}\right)\times[0,T), \tag{3.3}$$

$$\int_{\Sigma_i}\varepsilon\boldsymbol{E}\cdot\boldsymbol{n} = 0 \quad \text{in } [0,T), \quad i=1,\dots,I, \tag{3.4}$$

$$\boldsymbol{H}(\boldsymbol{x},0) = \boldsymbol{H}_0(\boldsymbol{x}) \quad \text{in } \mathbb{R}^3, \tag{3.5}$$

$$\boldsymbol{H}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \quad\text{and}\quad \boldsymbol{E}(\boldsymbol{x},t) = O\left(\frac{1}{|\boldsymbol{x}|}\right) \qquad \text{as } |\boldsymbol{x}|\to\infty, \tag{3.6}$$

where the asymptotic behavior (3.6) holds uniformly in $(0,T)$. The electric permittivity ε, the electric conductivity σ, and the magnetic permeability μ are piecewise smooth real-valued functions satisfying:

$$\varepsilon_1 \geq \varepsilon(\boldsymbol{x}) \geq \varepsilon_0 > 0 \quad \text{a.e. in } \Omega_{\rm c} \qquad\text{and}\qquad \varepsilon(\boldsymbol{x}) = \varepsilon_0 \quad \text{a.e. in } \mathbb{R}^3\setminus\Omega_{\rm c}, \tag{3.7}$$

$$\sigma_1 \geq \sigma(\boldsymbol{x}) \geq \sigma_0 > 0 \quad \text{a.e. in } \Omega_{\rm c} \qquad\text{and}\qquad \sigma(\boldsymbol{x}) = 0 \quad \text{a.e. in } \mathbb{R}^3\setminus\Omega_{\rm c}, \tag{3.8}$$

$$\mu_1 \geq \mu(\boldsymbol{x},t) \geq \mu_0 > 0 \quad \text{a.e. in } \Omega_{\rm c}\times[0,T) \quad \text{and} \quad \mu(\boldsymbol{x},t) = \mu_0 \quad \text{a.e. in } (\mathbb{R}^3\setminus\Omega_{\rm c})\times[0,T). \tag{3.9}$$

Notice that, as a consequence of (3.2) and (3.8), $\boldsymbol{J}$ must satisfy the compatibility conditions

$$\operatorname{div}\boldsymbol{J}(\boldsymbol{x},t) = 0 \quad \text{in } \mathbb{R}^3\setminus\Omega_{\rm c} \quad\text{and}\quad \int_{\Sigma_i}\boldsymbol{J}|_{\mathbb{R}^3\setminus\Omega_{\rm c}}\cdot\boldsymbol{n} = 0, \quad i=1,2,\dots,I. \tag{3.10}$$

We will formulate our problem in terms of the time primitive of the electric field

$$\boldsymbol{u}(\boldsymbol{x},t) := \int_0^t \boldsymbol{E}(\boldsymbol{x},s)\,ds.$$

To this end, we integrate (3.1) with respect to t,

$$\mu(\boldsymbol{x},t)\boldsymbol{H}(\boldsymbol{x},t) = -\,\mathbf{curl}\,\boldsymbol{u}(\boldsymbol{x},t) + \mu(\boldsymbol{x},0)\boldsymbol{H}_0, \tag{3.11}$$

and use the resulting expression of the magnetic field in (3.2) to obtain

$$\sigma\partial_t\boldsymbol{u} + \mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}\right) = \boldsymbol{f},$$

where

$$\boldsymbol{f}(\boldsymbol{x},t) := \mathbf{curl}\left(\frac{\mu(\boldsymbol{x},0)}{\mu(\boldsymbol{x},t)}\boldsymbol{H}_0\right) - \boldsymbol{J}(\boldsymbol{x},t). \tag{3.12}$$

It is important to remark that equation (3.2) provides, at the initial time $t=0$, the condition

$$\mathbf{curl}\,\boldsymbol{H}_0 = \boldsymbol{J}(\boldsymbol{x},0) + \sigma(\boldsymbol{x})\boldsymbol{E}(\boldsymbol{x},0) \quad \text{in } \mathbb{R}^3. \tag{3.13}$$

It then follows from our hypotheses on $\boldsymbol{J}$ and σ that the support of $\boldsymbol{f}$ is compact.

Notice that as a consequence of the decay conditions (3.6), we may assume that the electromagnetic field is weak far away from $\Omega_{\rm c}$. Motivated by this fact, and aiming to obtain a suitable simplification of our model problem, we introduce a closed surface Γ located sufficiently far from $\overline{\Omega}_{\rm c}$ and assume that $\boldsymbol{u}$ has a vanishing tangential trace on this surface. Hence, we will formulate our problem in the bounded domain Ω with boundary Γ. We assume that Ω is simply connected, with a connected boundary, and that it contains Ω_c and the support of $\boldsymbol{J}$ (and, consequently, the support of $\boldsymbol{f}$). We define $\Omega_d := \Omega \setminus \overline{\Omega}_c$.

The last considerations lead us to the following formulation of the eddy current problem:
Find $\boldsymbol{u} : \Omega \times [0,T) \to \mathbb{R}^3$ such that:

$$\begin{aligned}
\sigma\partial_t \boldsymbol{u} + \mathbf{curl}\left(\frac{1}{\mu}\,\mathbf{curl}\,\boldsymbol{u}\right) &= \boldsymbol{f} && \text{in } \Omega \times (0,T),\\
\operatorname{div}(\varepsilon \boldsymbol{u}) &= 0 && \text{in } \Omega_d \times [0,T),\\
\langle \gamma_{\boldsymbol{n}}(\varepsilon \boldsymbol{u}), 1\rangle_{\Sigma_i} &= 0 && \text{in } [0,T), \quad i = 1,\dots,I,\\
\gamma_\tau \boldsymbol{u} &= \mathbf{0} && \text{on } \Gamma \times [0,T),\\
\boldsymbol{u}(\cdot,0) &= \mathbf{0} && \text{in } \Omega.
\end{aligned} \tag{3.14}$$

We assume that both $\boldsymbol{J}$ and $\mathbf{curl}\left(\frac{\mu(\boldsymbol{x},0)}{\mu(\boldsymbol{x},t)}\boldsymbol{H}_0\right)$ belong to $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. Hence, we obtain a datum $\boldsymbol{f}$ for (3.14) that belongs to the same space. Besides, we deduce from (3.12) and (3.10) that $\boldsymbol{f}$ inherits from $\boldsymbol{J}$ the same compatibility conditions:

$$\operatorname{div} \boldsymbol{f} = 0 \text{ in } \Omega_d \qquad \text{and} \qquad \langle \gamma_{\boldsymbol{n}}(\boldsymbol{f}|_{\Omega_d}), 1\rangle_{\Sigma_i} = 0, \quad i = 1,2,\dots,I, \tag{3.15}$$

for all $t \in (0,T)$.

We introduce the space

$$M(\Omega_d) := \left\{\vartheta \in \mathrm{H}^1(\Omega_d) : \ \gamma\vartheta|_\Gamma = 0 \text{ and } \gamma\vartheta|_{\Sigma_i} = C_i, \ i = 1,\dots,I\right\},$$

where C_i, $i = 1,\dots,I$, are arbitrary constants. The Poincaré inequality shows that $|\cdot|_{1,\Omega_d}$ is a norm on $M(\Omega_d)$ equivalent to the usual $\mathrm{H}^1(\Omega_d)$-norm. Next, let

$$V_0(\Omega) := \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega) : \ b(\boldsymbol{v},\vartheta) = 0 \quad \forall \vartheta \in M(\Omega_d)\}, \tag{3.16}$$

where

$$b(\boldsymbol{u},\vartheta) := (\varepsilon \boldsymbol{u}, \mathbf{grad}\,\vartheta)_{0,\Omega_d}.$$

Lemma 3.1. *It follows that*

$$V_0(\Omega) = \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl},\Omega) : \ \operatorname{div}(\varepsilon\boldsymbol{v}) = 0 \textit{ in } \Omega_d, \ \langle \gamma_{\boldsymbol{n}}(\varepsilon\boldsymbol{v}), 1\rangle_{\Sigma_i} = 0, \ i = 1,\dots,I\}.$$

Proof. If $\boldsymbol{v} \in V_0(\Omega)$, then, in particular,

$$b(\boldsymbol{v},\vartheta) = 0 \qquad \forall \vartheta \in \mathcal{D}(\Omega_d),$$

where $\mathcal{D}(\Omega_d)$ is the space of infinitely differentiable functions with compact support in Ω_d. This implies that $\operatorname{div}(\varepsilon\boldsymbol{v}) = 0$ in Ω_d. Now choosing $\vartheta_i \in M(\Omega_d)$ such that $\gamma\vartheta_i|_{\Sigma_j} = \delta_{i,j}$ for $1 \le i,j \le I$, we obtain from a Green's formula that

$$0 = b(\boldsymbol{v},\vartheta_i) = \langle \gamma_{\boldsymbol{n}}(\varepsilon\boldsymbol{v}), 1\rangle_{\Sigma_i}.$$

The other inclusion is straightforward. □

By testing the first equation of (3.14) with a function $\boldsymbol{v} \in V_0(\Omega)$ and using (2.1), we obtain the following variational formulation:

Find $\boldsymbol{u} \in \mathcal{W}_0$ such that

(3.17)
$$\frac{d}{dt}(\sigma \boldsymbol{u}(t), \boldsymbol{v})_{0,\Omega_{\mathrm{c}}} + \left(\frac{1}{\mu(t)} \operatorname{\mathbf{curl}} \boldsymbol{u}(t), \operatorname{\mathbf{curl}} \boldsymbol{v}\right)_{0,\Omega} = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in V_0(\Omega),$$
$$\boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0) = \mathbf{0},$$

where

$$\mathcal{W}_0 := \{\boldsymbol{v} \in \mathrm{L}^2(0,T; V_0(\Omega)) : \ \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in W^1(0,T; \mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}}))\},$$

with

$$W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})) := \left\{\boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})) : \ \partial_t \boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})')\right\}.$$

Here, $\mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})'$ is the dual space of $\mathbf{H}(\mathbf{curl}, \Omega_{\mathrm{c}})$ with respect to the pivot space

$$\mathrm{L}^2(\Omega_{\mathrm{c}}, \sigma)^3 := \left\{\boldsymbol{v} : \Omega_{\mathrm{c}} \to \mathbb{R}^3 \text{ Lebesgue-measurable} : \ \int_{\Omega_{\mathrm{c}}} \sigma |\boldsymbol{v}|^2 < \infty\right\}.$$

Notice that the initial condition makes sense thanks to the continuous embedding $W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})) \hookrightarrow \mathcal{C}^0(0,T;\mathrm{L}^2(\Omega_{\mathrm{c}},\sigma)^3)$; see for instance [24, Proposition 23.23].

In order to avoid the task of constructing a conforming finite element discretization of (3.17), we take advantage of Lemma 3.1 and propose a mixed formulation of the problem. To this end, we relax as follows the divergence-free restriction through a Lagrange multiplier:
Find $\boldsymbol{u} \in \mathcal{W}$ and $\lambda \in \mathrm{L}^2(0,T; M(\Omega_{\mathrm{d}}))$ such that

(3.18)
$$\frac{d}{dt}\left[(\boldsymbol{u}(t), \boldsymbol{v})_\sigma + b(\boldsymbol{v}, \lambda(t))\right] + a(t; \boldsymbol{u}(t), \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega),$$
$$b(\boldsymbol{u}(t), \vartheta) = 0 \qquad \forall \vartheta \in M(\Omega_{\mathrm{d}}),$$
$$\boldsymbol{u}|_{\Omega_{\mathrm{c}}}(0) = \mathbf{0},$$

where

$$\mathcal{W} := \left\{\boldsymbol{v} \in \mathrm{L}^2(0,T;\mathbf{H}_0(\mathbf{curl},\Omega)) : \ \boldsymbol{v}|_{\Omega_{\mathrm{c}}} \in W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}))\right\},$$

$$(\boldsymbol{u}, \boldsymbol{v})_\sigma := (\sigma \boldsymbol{u}, \boldsymbol{v})_{0,\Omega_{\mathrm{c}}} \qquad \text{and} \qquad a(t; \boldsymbol{u}, \boldsymbol{v}) := \left(\frac{1}{\mu(t)} \operatorname{\mathbf{curl}} \boldsymbol{u}, \operatorname{\mathbf{curl}} \boldsymbol{v}\right)_{0,\Omega}.$$

Notice that $\mathcal{W}$, endowed with the graph norm

$$\|\boldsymbol{v}\|_{\mathcal{W}}^2 := \int_0^T \|\boldsymbol{v}(t)\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 \, dt + \int_0^T \|\partial_t \boldsymbol{v}(t)\|_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})'}^2 \, dt,$$

is a Hilbert space and that $\mathcal{W}_0$ is a closed subspace of $\mathcal{W}$.

4. Existence and uniqueness

We introduce the space

$$V_0(\Omega_{\mathrm{d}}) := \{\boldsymbol{v} \in \mathbf{H}_0(\mathbf{curl}, \Omega_{\mathrm{d}}) : \ b(\boldsymbol{v}, \vartheta) = 0 \quad \forall \vartheta \in M(\Omega_{\mathrm{d}})\}$$

and recall the following result.

Lemma 4.1. *The seminorm* $\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\rm d}}$ *is a norm on* $V_0(\Omega_{\rm d})$ *equivalent to the usual norm of* $\mathbf{H}(\mathbf{curl},\Omega_{\rm d})$*; i.e., there exists a constant* $C>0$ *depending only on* Ω *such that*

$$\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C\|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\rm d}} \qquad \forall \boldsymbol{v}\in V_0(\Omega_{\rm d}).$$

Proof. See, for instance, [14, Corollary 4.4]. □

Lemma 4.2. *The linear mapping*

$$\begin{array}{rcl}\mathcal{E}:\ \mathbf{H}(\mathbf{curl},\Omega_{\rm c}) & \to & V_0(\Omega)\\ \boldsymbol{v}_{\rm c} & \mapsto & \mathcal{E}\boldsymbol{v}_{\rm c}\end{array}$$

characterized by $(\mathcal{E}\boldsymbol{v}_{\rm c})|_{\Omega_{\rm c}} = \boldsymbol{v}_{\rm c}$ *and*

$$(\mathbf{curl}\,\mathcal{E}\boldsymbol{v}_{\rm c},\mathbf{curl}\,\boldsymbol{w})_{0,\Omega_{\rm d}} = 0 \qquad \forall \boldsymbol{w}\in V_0(\Omega_{\rm d}) \tag{4.1}$$

is well defined and bounded.

Proof. Let us denote here by $\gamma_\tau^{\rm c}$ and $\gamma_\tau^{\rm d}$ the tangential traces on Σ taken from $\Omega_{\rm c}$ and $\Omega_{\rm d}$, respectively. We know from Theorem 2.1 that there exists a continuous right inverse of the tangential trace operator $\gamma_\tau^{\rm d}$:

$$(\gamma_\tau^{\rm d})^{-1}:\ \mathbf{H}^{-1/2}(\operatorname{div}_\Gamma,\Sigma)\to\{\boldsymbol{v}|_{\Omega_{\rm d}}:\ \boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega)\}.$$

It follows that the linear operator

$$\begin{array}{rcl}\mathcal{L}:\ \mathbf{H}(\mathbf{curl},\Omega_{\rm c}) & \to & \{\boldsymbol{v}|_{\Omega_{\rm d}}:\ \boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega)\}\\ \boldsymbol{v}_{\rm c} & \mapsto & \mathcal{L}\boldsymbol{v}_{\rm c} := (\gamma_\tau^{\rm d})^{-1}(\gamma_\tau^{\rm c}\boldsymbol{v}_{\rm c})\end{array} \tag{4.2}$$

is bounded, namely,

$$\|\mathcal{L}\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C_0\|\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm c})} \qquad \forall \boldsymbol{v}_{\rm c}\in\mathbf{H}(\mathbf{curl},\Omega_{\rm c}), \tag{4.3}$$

and it satisfies $\gamma_\tau^{\rm d}(\mathcal{L}\boldsymbol{v}_{\rm c}) = \gamma_\tau^{\rm c}\boldsymbol{v}_{\rm c}$ on Σ. Notice that $\mathcal{L}\boldsymbol{v}_{\rm c}$ is an $\mathbf{H}(\mathbf{curl},\Omega)$-conforming extension of $\boldsymbol{v}_{\rm c}$ to the whole Ω, but it does not necessarily fulfill (4.1).

Given $\boldsymbol{v}_{\rm c}\in\mathbf{H}(\mathbf{curl},\Omega_{\rm c})$, consider the problem of finding $\boldsymbol{z}\in\mathcal{L}\boldsymbol{v}_{\rm c}+\mathbf{H}_0(\mathbf{curl},\Omega_{\rm d})$ and $\rho\in M(\Omega_{\rm d})$ satisfying

$$\begin{aligned}(\mathbf{curl}\,\boldsymbol{z},\mathbf{curl}\,\boldsymbol{w})_{0,\Omega_{\rm d}} + b(\boldsymbol{w},\rho) &= 0 \qquad \forall \boldsymbol{w}\in\mathbf{H}_0(\mathbf{curl},\Omega_{\rm d}),\\ b(\boldsymbol{z},\vartheta) &= 0 \qquad \forall \vartheta\in M(\Omega_{\rm d}).\end{aligned}$$

The well-posedness of this problem is guaranteed by the Babuška-Brezzi theory. Indeed, on the one hand, the fact that $\mathbf{grad}(M(\Omega_{\rm d}))\subset\mathbf{H}_0(\mathbf{curl},\Omega_{\rm d})$ implies easily the following inf-sup condition for b:

$$\sup_{\boldsymbol{z}\in\mathbf{H}_0(\mathbf{curl},\Omega_{\rm d})}\frac{b(\boldsymbol{z},\vartheta)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} \ge \varepsilon_0\frac{(\mathbf{grad}\,\vartheta,\mathbf{grad}\,\vartheta)_{0,\Omega_{\rm d}}}{\|\mathbf{grad}\,\vartheta\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} = \varepsilon_0|\vartheta|_{1,\Omega_{\rm d}} \qquad \forall\vartheta\in M(\Omega_{\rm d}).$$

On the other hand, Lemma 4.1 ensures the ellipticity in the kernel property: there exists $C_1>0$ such that

$$(\mathbf{curl}\,\boldsymbol{w},\mathbf{curl}\,\boldsymbol{w})_{0,\Omega_{\rm d}} \ge C_1\|\boldsymbol{w}\|^2_{\mathbf{H}_0(\mathbf{curl},\Omega_{\rm d})} \qquad \forall \boldsymbol{w}\in V_0(\Omega_{\rm d}). \tag{4.4}$$

It is now clear that $\mathcal{E}\boldsymbol{v}_{\rm c} := \boldsymbol{z}$ satisfies (4.1) and $(\mathcal{E}\boldsymbol{v}_{\rm c})|_{\Omega_{\rm c}} = \boldsymbol{v}_{\rm c}$. The uniqueness of the solution of (4.1) follows from (4.4). Moreover, by virtue of the stability results provided by the Babuška-Brezzi theory, there exists a constant $C_2>0$ such that

$$\|\mathcal{E}\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C_2\|\mathcal{L}\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}.$$

Finally, (4.3) yields the estimate

$$\|\mathcal{E}\boldsymbol{v}_{\mathrm{c}}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \le \sqrt{1+(C_0C_2)^2}\|\boldsymbol{v}_{\mathrm{c}}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})} \qquad \forall \boldsymbol{v}_{\mathrm{c}} \in \mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}). \qquad \square$$

Lemma 4.3. *The inner product*

$$(\boldsymbol{u},\boldsymbol{v})_{V_0(\Omega)} := (\boldsymbol{u},\boldsymbol{v})_\sigma + (\mathbf{curl}\,\boldsymbol{u},\mathbf{curl}\,\boldsymbol{v})_{0,\Omega} \tag{4.5}$$

induces in $V_0(\Omega)$ *a norm* $\|\cdot\|_{V_0(\Omega)}$ *that is equivalent to the* $\mathbf{H}(\mathbf{curl},\Omega)$ *norm. Moreover, the following decomposition is orthogonal with respect to the inner product* $(\cdot,\cdot)_{V_0(\Omega)}$*:*

$$V_0(\Omega) = \widetilde{V_0(\Omega_{\mathrm{d}})} \oplus \mathcal{E}(\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})), \tag{4.6}$$

where $\widetilde{V_0(\Omega_{\mathrm{d}})}$ *is the subspace of* $V_0(\Omega)$ *obtained by extending by zero the functions of* $V_0(\Omega_{\mathrm{d}})$ *to the whole domain* Ω*.*

Proof. For any $\boldsymbol{v} \in V_0(\Omega)$, let us use the notation $\boldsymbol{v}_{\mathrm{c}} := \boldsymbol{v}|_{\Omega_{\mathrm{c}}}$. Notice that $\boldsymbol{v}-\mathcal{E}\boldsymbol{v}_{\mathrm{c}} \in \widetilde{V_0(\Omega_{\mathrm{d}})}$. The triangle inequality and Lemma 4.1 ensure the existence of a constant $C_0 > 0$ such that

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \le 2C_0^2\|\mathbf{curl}(\boldsymbol{v}-\mathcal{E}\boldsymbol{v}_{\mathrm{c}})\|^2_{0,\Omega_{\mathrm{d}}} + 2\|\mathcal{E}\boldsymbol{v}_{\mathrm{c}}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}.$$

Hence, using again the triangle inequality and Lemma 4.2, we have

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \le C_1\left(\|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega_{\mathrm{d}}} + \|\boldsymbol{v}_{\mathrm{c}}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}\right) = C_1\left(\|\boldsymbol{v}\|^2_{0,\Omega_{\mathrm{c}}} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{0,\Omega}\right).$$

Consequently,

$$\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \le C_1\max\{\sigma_0^{-1},1\}\,\|\boldsymbol{v}\|^2_{V_0(\Omega)}.$$

The other inequality is straightforward.

Finally, it is easy to check that $\mathcal{E}(\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}))$ is the orthogonal complement of $\widetilde{V_0(\Omega_{\mathrm{d}})}$ in $V_0(\Omega)$ with respect to the inner product $(\cdot,\cdot)_{V_0(\Omega)}$. $\square$

We are now in a position to prove the main result of this section.

Theorem 4.4. *Problem* (3.18) *has a unique solution* $(\boldsymbol{u},\lambda)$*. Furthermore, there exists* $C > 0$ *such that*

$$\max_{t\in[0,T]}\|\boldsymbol{u}(t)\|^2_{0,\Omega_{\mathrm{c}}} + \int_0^T \|\boldsymbol{u}(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}\,dt \le C\int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt. \tag{4.7}$$

Proof. We first notice that the second equation of (3.18) means that $\boldsymbol{u} \in \mathcal{W}_0$. The decomposition (4.6) implies that the direct sum

$$\mathcal{W}_0 = \mathrm{L}^2(0,T;\widetilde{V_0(\Omega_{\mathrm{d}})}) \oplus \mathcal{E}(W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})))$$

is orthogonal with respect to the inner product $\int_0^T(\cdot,\cdot)_{V_0(\Omega)}\,dt$. Hence $\boldsymbol{u} = \boldsymbol{u}_{\mathrm{d}} + \mathcal{E}\boldsymbol{u}_{\mathrm{c}}$, with $\boldsymbol{u}_{\mathrm{d}} \in \mathrm{L}^2(0,T;\widetilde{V_0(\Omega_{\mathrm{d}})})$ and $\mathcal{E}\boldsymbol{u}_{\mathrm{c}} \in \mathcal{E}(W^1(0,T;\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})))$. Testing the first equation of (3.18) with $\boldsymbol{v} \in \widetilde{V_0(\Omega_{\mathrm{d}})}$, we find that the first component satisfies

$$\left(\frac{1}{\mu(t)}\mathbf{curl}\,\boldsymbol{u}_{\mathrm{d}}(t),\mathbf{curl}\,\boldsymbol{v}\right)_{0,\Omega_{\mathrm{d}}} = (\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega_{\mathrm{d}}} \qquad \forall \boldsymbol{v} \in V_0(\Omega_{\mathrm{d}}). \tag{4.8}$$

Lemma 4.1 and the Lax-Milgram lemma prove that this problem admits a unique solution and that there exists $C_1 > 0$ such that

$$\int_0^T \|\boldsymbol{u}_{\mathrm{d}}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{d}})}\,dt \le C_1\int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt. \tag{4.9}$$

The other component is determined by solving

$$\frac{d}{dt}(\boldsymbol{u}_{\mathrm{c}}(t),\boldsymbol{v})_\sigma + a(t;\mathcal{E}\boldsymbol{u}_{\mathrm{c}}(t),\mathcal{E}\boldsymbol{v}) = (\boldsymbol{f}(t),\mathcal{E}\boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v}\in \mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}), \quad \boldsymbol{u}_{\mathrm{c}}(0)=\mathbf{0}. \tag{4.10}$$

For any $t\in(0,T)$, the bilinear form $a(t;\mathcal{E}\cdot,\mathcal{E}\cdot)$ is clearly continuous and coercive on $\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})$:

$$a(t;\mathcal{E}\boldsymbol{v},\mathcal{E}\boldsymbol{v}) + (\boldsymbol{v},\boldsymbol{v})_\sigma \geq \min\{\sigma_0,\mu_1^{-1}\}\|\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}})} \qquad \forall \boldsymbol{v}\in \mathbf{H}(\mathbf{curl},\Omega_{\mathrm{c}}).$$

Therefore, the well-posedness of the parabolic problem (4.10) follows immediately from a simple variant of the Lions theorem (see, for instance, [24, Corollary 23.26]). In addition, there exists $C_2>0$ such that

$$\max_{t\in[0,T]}\|\boldsymbol{u}_{\mathrm{c}}(t)\|^2_{0,\Omega_{\mathrm{c}}} + \int_0^T \|\boldsymbol{u}_{\mathrm{c}}(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega_{\mathrm{d}})}\,dt \leq C_2\int_0^T \|\boldsymbol{f}(t)\|^2_{0,\Omega}\,dt,$$

which, combined with (4.9) and the boundedness of $\mathcal{E}$, yields (4.7).

It remains to prove the existence and uniqueness of the Lagrange multiplier λ. Given $\vartheta\in M(\Omega_{\mathrm{d}})$, we denote by $\widetilde{\mathbf{grad}}\,\vartheta\in\mathbf{H}_0(\mathbf{curl},\Omega)$ the extension by zero of $\mathbf{grad}\,\vartheta$ to the whole Ω. Notice that the bilinear form b satisfies the inf-sup condition

$$\sup_{\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega)}\frac{b(\boldsymbol{v},\vartheta)}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}} \geq \frac{b(\widetilde{\mathbf{grad}}\,\vartheta,\vartheta)}{\|\widetilde{\mathbf{grad}}\,\vartheta\|_{\mathbf{H}(\mathbf{curl},\Omega)}} = \varepsilon_0|\vartheta|_{1,\Omega_{\mathrm{d}}} \qquad \forall\vartheta\in M(\Omega_{\mathrm{d}}). \tag{4.11}$$

Let us now consider $\mathcal{G}\in\mathcal{C}^0([0,T],\mathbf{H}_0(\mathbf{curl},\Omega)')$ defined by

$$\langle\mathcal{G}(t),\boldsymbol{v}\rangle := -\,(\boldsymbol{u}(t),\boldsymbol{v})_\sigma - \int_0^t a(s;\boldsymbol{u}(s),\boldsymbol{v})\,ds + \int_0^t (\boldsymbol{f}(s),\boldsymbol{v})_{0,\Omega}\,ds$$

for all $\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega)$. By integrating the first equation of (3.17) with respect to t and using the second one, we obtain

$$\langle\mathcal{G}(t),\boldsymbol{v}\rangle = 0 \qquad \forall\boldsymbol{v}\in V_0(\Omega).$$

Therefore, taking into account the definition (3.16) of $V_0(\Omega)$, the inf-sup condition (4.11) guarantees the existence of a unique $\lambda(t)\in M(\Omega_{\mathrm{d}})$ such that (see [13, Lemma I.4.1])

$$b(\boldsymbol{v},\lambda(t)) = \langle\mathcal{G}(t),\boldsymbol{v}\rangle \qquad \forall\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega). \tag{4.12}$$

We conclude that $(\boldsymbol{u},\lambda)$ solves (3.18) by differentiating the last identity with respect to t in the sense of distributions. □

The reason for which we have skipped the stability estimate for the Lagrange multiplier λ in the last theorem becomes clear from the following result.

Lemma 4.5. *The Lagrange multiplier λ of problem* (3.18) *vanishes identically.*

Proof. By virtue of the compatibility conditions (3.15),

$$(\boldsymbol{f},\mathbf{grad}\,\vartheta)_{0,\Omega_{\mathrm{d}}} = \langle\gamma_{\boldsymbol{n}}\boldsymbol{f},\vartheta\rangle_{\partial\Omega_{\mathrm{d}}} = \sum_{i=1}^{I}\vartheta|_{\Sigma_i}\langle\gamma_{\boldsymbol{n}}\boldsymbol{f},1\rangle_{\Sigma_i} = 0 \qquad \forall\vartheta\in M(\Omega_{\mathrm{d}}). \tag{4.13}$$

Consequently, testing the first equation of (3.18) with $\mathbf{grad}\,\vartheta$ (extended by zero to the whole Ω) yields

$$\frac{d}{dt}b(\mathbf{grad}\,\vartheta,\lambda(t)) = (\boldsymbol{f}(t),\mathbf{grad}\,\vartheta)_{0,\Omega_{\mathrm{d}}} = 0 \qquad \forall\vartheta\in M(\Omega_{\mathrm{d}}).$$

Next, we take $t=0$ in (4.12) and use the fact that $\mathcal{G}(0)=\mathbf{0}$ to deduce that $t\mapsto b(\mathbf{grad}\,\vartheta,\lambda(t))$ vanishes identically in $[0,T]$ for all $\vartheta\in M(\Omega_{\mathrm{d}})$. In particular, $\varepsilon_0|\lambda(t)|^2_{1,\Omega_{\mathrm{d}}} = b(\mathbf{grad}\,\lambda(t),\lambda(t)) = 0$ for all $t\in[0,T]$, and the result follows. □

Remark 4.6. As a consequence of (3.13), we have that $\boldsymbol{f}(\boldsymbol{x},0) := \mathbf{curl}\,\boldsymbol{H}_0 - \boldsymbol{J}(\boldsymbol{x},0) = \mathbf{0}$. Now, solving (4.8) at $t=0$ shows that $\boldsymbol{u}_{\mathrm{d}}(\boldsymbol{x},0)=0$ in Ω_{d}. This proves that the global initial condition

$$\boldsymbol{u}(\boldsymbol{x},0)=\mathbf{0} \quad \text{in } \Omega$$

of problem (3.14) holds true.

5. Analysis of the semi-discrete scheme

In what follows we assume that Ω and Ω_{c} are Lipschitz polyhedra. Let $\{\mathcal{T}_h\}_h$ be a regular family of tetrahedral meshes of Ω such that each element $K\in\mathcal{T}_h$ is contained either in $\overline{\Omega}_{\mathrm{c}}$ or in $\overline{\Omega}_{\mathrm{d}}$. As usual, h stands for the largest diameter of the tetrahedra K in $\mathcal{T}_h$. Furthermore, we suppose that the family of triangulations $\{\mathcal{T}_h(\Sigma)\}_h$ induced by $\{\mathcal{T}_h\}_h$ on Σ is quasi-uniform.

We define a semidiscrete version of (3.18) by means of Nédélec finite elements. The local representation of the mth-order element of this family on a tetrahedron K is given by (see [19, Section 5.5])

$$\mathcal{N}_m(K) := \mathbb{P}^3_{m-1}\oplus S_m,$$

where $\mathbb{P}_m$ is the set of polynomials of degree not greater than m and

$$S_m := \left\{p\in\widetilde{\mathbb{P}}^3_m:\ \boldsymbol{x}\cdot p(\boldsymbol{x})=0\right\},$$

with $\widetilde{\mathbb{P}}_m$ being the set of homogeneous polynomials of degree m. The corresponding global space $X_h(\Omega)$ is the space of functions that are locally in $\mathcal{N}_m(K)$ and have continuous tangential components across the faces of the triangulation $\mathcal{T}_h$:

$$X_h(\Omega) := \{\boldsymbol{v}\in\mathbf{H}_0(\mathbf{curl},\Omega):\ \boldsymbol{v}|_K\in\mathcal{N}_m(K)\ \forall K\in\mathcal{T}_h\}.$$

We use standard mth-order Lagrange finite elements to approximate $M(\Omega_{\mathrm{d}})$:

$$M_h(\Omega_{\mathrm{d}}) := \left\{\vartheta\in\mathrm{H}^1(\Omega_{\mathrm{d}}):\ \vartheta|_K\in\mathbb{P}_m\ \forall K\in\mathcal{T}_h,\ \vartheta|_\Gamma=0,\ \vartheta|_{\Sigma_i}=C_i,\ i=1,\dots,I\right\}.$$

We introduce the following semi-discretization of problem (3.18):
Find $\boldsymbol{u}_h(t):[0,T]\to X_h(\Omega)$ and $\lambda_h(t):[0,T]\to M_h(\Omega_{\mathrm{d}})$ such that

$$\begin{aligned}
&\frac{d}{dt}\left[(\boldsymbol{u}_h(t),\boldsymbol{v})_\sigma + b(\boldsymbol{v},\lambda_h(t))\right] + a(t;\boldsymbol{u}_h(t),\boldsymbol{v}) = (\boldsymbol{f}(t),\boldsymbol{v})_{0,\Omega} \qquad \forall\boldsymbol{v}\in X_h(\Omega),\\
&b(\boldsymbol{u}_h(t),\vartheta)=0 \qquad \forall\vartheta\in M_h(\Omega_{\mathrm{d}}),\\
&\boldsymbol{u}_h|_{\Omega_{\mathrm{c}}}(0)=\mathbf{0}.
\end{aligned}\tag{5.1}$$

Notice that the discrete kernel

$$V_{0,h}(\Omega) := \{\boldsymbol{v}\in X_h(\Omega):\ b(\boldsymbol{v},\vartheta)=0\ \forall\vartheta\in M_h(\Omega_{\mathrm{d}})\}$$

is not necessarily a subspace of $V_0(\Omega)$. We introduce

$$V_{0,h}(\Omega_{\rm d}) := \{\boldsymbol{v}|_{\Omega_{\rm d}} :\ \boldsymbol{v} \in V_{0,h}(\Omega)\} \cap \mathbf{H}_0(\mathbf{curl}, \Omega_{\rm d})$$

and recall the discrete analogue of Lemma 4.1.

Lemma 5.1. *The mapping* $\boldsymbol{v} \mapsto \|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\rm d}}$ *is a norm on* $V_{0,h}(\Omega_{\rm d})$ *uniformly equivalent to the* $\mathbf{H}(\mathbf{curl}, \Omega_{\rm d})$*-norm; i.e., there exists* $C > 0$*, independent of* h*, such that*

$$\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C\|\mathbf{curl}\,\boldsymbol{v}\|_{0,\Omega_{\rm d}} \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega_{\rm d}). \tag{5.2}$$

Proof. See, for instance, [14, Theorem 4.7]. □

We will also need the following result deduced from Proposition 3.3 of [2], which makes use of the quasi-uniformity of $\{\mathcal{T}_h(\Sigma)\}_h$.

Lemma 5.2. *Let*

$$X_h(\Omega_{\rm c}) := \{\boldsymbol{v}|_{\Omega_{\rm c}} :\ \boldsymbol{v} \in X_h(\Omega)\} \quad \text{and} \quad X_h(\Omega_{\rm d}) := \{\boldsymbol{v}|_{\Omega_{\rm d}} :\ \boldsymbol{v} \in X_h(\Omega)\}.$$

There exists a linear operator

$$\mathcal{F}_h :\ \gamma_\tau(X_h(\Omega_{\rm c})) \to X_h(\Omega_{\rm d})$$

such that $\gamma_\tau(\mathcal{F}_h \boldsymbol{\eta}_h) = \boldsymbol{\eta}_h$ *and*

$$\|\mathcal{F}_h\boldsymbol{\eta}_h\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C\|\boldsymbol{\eta}_h\|_{\mathbf{H}^{-1/2}(\mathrm{div}_\Gamma,\Gamma)} \quad \forall \boldsymbol{\eta}_h \in \gamma_\tau(X_h(\Omega_{\rm c})),$$

for some positive constant C *independent of* h*.*

Lemma 5.3. *The linear mapping*

$$\begin{array}{rccc} \mathcal{E}_h : & X_h(\Omega_{\rm c}) & \to & V_{0,h}(\Omega) \\ & \boldsymbol{v}_{\rm c} & \mapsto & \mathcal{E}_h\boldsymbol{v}_{\rm c} \end{array}$$

characterized by $(\mathcal{E}_h\boldsymbol{v}_{\rm c})|_{\Omega_{\rm c}} = \boldsymbol{v}_{\rm c}$ *and*

$$(\mathbf{curl}\,\mathcal{E}_h\boldsymbol{v}_{\rm c}, \mathbf{curl}\,\boldsymbol{w})_{0,\Omega_{\rm d}} = 0 \qquad \forall \boldsymbol{w} \in V_{0,h}(\Omega_{\rm d}) \tag{5.3}$$

is well defined and bounded uniformly in h*.*

Proof. Combining Theorem 2.1 and Lemma 5.2, we deduce that the linear mapping $\mathcal{L}_h :\ X_h(\Omega_{\rm c}) \to X_h(\Omega_{\rm d})$ given by $\mathcal{L}_h\boldsymbol{v}_{\rm c} = \mathcal{F}_h(\gamma_\tau\boldsymbol{v}_{\rm c})$ is uniformly bounded; namely, there exists $C_0 > 0$, independent of h, such that

$$\|\mathcal{L}_h\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C_0\|\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm c})} \qquad \forall \boldsymbol{v}_{\rm c} \in X_h(\Omega_{\rm c}). \tag{5.4}$$

The mixed version of (5.3) consists of finding $\boldsymbol{z}_h \in \mathcal{L}_h\boldsymbol{v}_{\rm c} + X_{0,h}(\Omega_{\rm d})$ and $\rho_h \in M_h(\Omega_{\rm d})$ such that

$$\begin{aligned} (\mathbf{curl}\,\boldsymbol{z}_h, \mathbf{curl}\,\boldsymbol{w})_{0,\Omega_{\rm d}} + b(\boldsymbol{w}, \rho_h) &= 0 \qquad \forall \boldsymbol{w} \in X_{0,h}(\Omega_{\rm d}), \\ b(\boldsymbol{z}_h, \vartheta) &= 0 \qquad \forall \vartheta \in M_h(\Omega_{\rm d}), \end{aligned}$$

where $X_{0,h}(\Omega_{\rm d}) := X_h(\Omega_{\rm d}) \cap \mathbf{H}_0(\mathbf{curl}, \Omega_{\rm d})$. Similarly to the continuous case, $\mathbf{grad}(M_h(\Omega_{\rm d})) \subset X_{0,h}(\Omega_{\rm d})$ and hence

$$\sup_{\boldsymbol{z}\in X_{0,h}(\Omega_{\rm d})} \frac{b(\boldsymbol{z}, \vartheta)}{\|\boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} \ge \varepsilon_0 \frac{(\mathbf{grad}\,\vartheta, \mathbf{grad}\,\vartheta)_{0,\Omega_{\rm d}}}{\|\,\mathbf{grad}\,\vartheta\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})}} = \varepsilon_0|\vartheta|_{1,\Omega_{\rm d}} \qquad \forall \vartheta \in M_h(\Omega_{\rm d}).$$

This discrete inf-sup condition and (5.2) allow us to apply again the Babuška-Brezzi theory to deduce that $\boldsymbol{z}_h$ is well defined and

$$\|\boldsymbol{z}_h\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm d})} \le C_1\|\mathcal{L}_h\boldsymbol{v}_{\rm c}\|_{\mathbf{H}(\mathbf{curl},\Omega_{\rm c})},$$

with $C_1 > 0$ independent of h. If we define $\mathcal{E}_h \boldsymbol{v}_\mathrm{c} := \boldsymbol{z}_h$, clearly $(\mathcal{E}_h \boldsymbol{v}_\mathrm{c})|_{\Omega_\mathrm{c}} = \boldsymbol{v}_\mathrm{c}$ and (5.3) holds true. Moreover, these two conditions clearly determine $\mathcal{E}_h \boldsymbol{v}_\mathrm{c}$ uniquely, and applying (5.4) we have

$$\|\mathcal{E}_h \boldsymbol{v}_\mathrm{c}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq \sqrt{1 + (C_0 C_1)^2} \|\boldsymbol{v}_\mathrm{c}\|_{\mathbf{H}(\mathbf{curl},\Omega_\mathrm{c})} \qquad \forall \boldsymbol{v}_\mathrm{c} \in \mathbf{H}(\mathbf{curl},\Omega),$$

from which the result follows. □

Proceeding exactly as in the previous section we obtain the following result.

Lemma 5.4. *The bilinear form $(\cdot,\cdot)_{V_0(\Omega)}$ induces a norm on $V_{0,h}(\Omega)$ uniformly equivalent to the $\mathbf{H}(\mathbf{curl},\Omega)$-norm; i.e., there exists $C_1 > 0$ and $C_2 > 0$, independent of h, such that*

$$C_1 \|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq \|\boldsymbol{v}\|_{V_0(\Omega)} \leq C_2 \|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega).$$

Moreover, we have the following $(\cdot,\cdot)_{V_0(\Omega)}$-orthogonal decomposition:

$$V_{0,h}(\Omega) = \widetilde{V_{0,h}(\Omega_\mathrm{d})} \oplus \mathcal{E}_h(X_h(\Omega_\mathrm{c})), \tag{5.5}$$

where $\widetilde{V_{0,h}(\Omega_\mathrm{d})}$ is the subspace of $V_{0,h}(\Omega)$ obtained by extending the functions of $V_{0,h}(\Omega_\mathrm{d})$ by zero to the whole domain Ω.

Theorem 5.5. *Problem* (5.1) *has a unique solution $(\boldsymbol{u}_h, \lambda_h)$ with an identically vanishing discrete Lagrange multiplier λ_h.*

Proof. According to (5.5), we look for a solution of problem (5.1) written as follows: $\boldsymbol{u}_h = \boldsymbol{u}_{\mathrm{d},h} + \mathcal{E}_h(\boldsymbol{u}_{\mathrm{c},h})$, with $\boldsymbol{u}_{\mathrm{d},h}(t) \in \widetilde{V_{0,h}(\Omega_\mathrm{d})}$ and $\boldsymbol{u}_{\mathrm{c},h} \in X_h(\Omega_\mathrm{c})$. Notice that $\boldsymbol{u}_{\mathrm{d},h}(t)|_{\Omega_\mathrm{d}} \in V_{0,h}(\Omega_\mathrm{d})$ must be the unique solution of the problem

$$\left(\frac{1}{\mu(t)} \mathbf{curl}\, \boldsymbol{u}_{\mathrm{d},h}(t), \mathbf{curl}\, \boldsymbol{v}\right)_{0,\Omega_\mathrm{d}} = (\boldsymbol{f}(t), \boldsymbol{v})_{0,\Omega_\mathrm{d}} \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega_\mathrm{d}).$$

The other term $\boldsymbol{u}_{\mathrm{c},h}$ has to be the unique solution of the finite-dimensional initial value problem

$$\frac{d}{dt} (\boldsymbol{u}_{\mathrm{c},h}(t), \boldsymbol{v})_\sigma + a(t; \mathcal{E}_h \boldsymbol{u}_{\mathrm{c},h}(t), \mathcal{E}_h \boldsymbol{v}) = (\boldsymbol{f}(t), \mathcal{E}_h \boldsymbol{v})_{0,\Omega} \qquad \forall \boldsymbol{v} \in X_h(\Omega_\mathrm{c}),$$
$$\boldsymbol{u}_{\mathrm{c},h}(0) = \mathbf{0}.$$

It only remains to prove the existence and uniqueness of the Lagrange multiplier λ_h. With this aim we notice that the functional defined by

$$\langle \mathcal{G}_h(t), \boldsymbol{v} \rangle := \int_0^t \Big[(\boldsymbol{f}(s), \boldsymbol{v})_{0,\Omega} - a(s; \boldsymbol{u}_h(s), \boldsymbol{v}) \Big] \, ds - (\boldsymbol{u}_h(t), \boldsymbol{v})_\sigma$$

vanishes on the discrete kernel:

$$\langle \mathcal{G}_h(t), \boldsymbol{v} \rangle = 0 \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega).$$

Hence, the discrete inf-sup condition,

$$\sup_{\boldsymbol{v} \in X_h(\Omega)} \frac{b(\boldsymbol{v}, \vartheta)}{\|\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}} \geq \frac{b(\widetilde{\mathbf{grad}}\, \vartheta, \vartheta)}{\|\widetilde{\mathbf{grad}}\, \vartheta\|_{\mathbf{H}(\mathbf{curl},\Omega)}} = \varepsilon_0 |\vartheta|_{1,\Omega_\mathrm{d}} \qquad \forall \vartheta \in M_h(\Omega_\mathrm{d}),$$

implies that there exists a unique $\lambda_h(t)$ satisfying

$$b(\boldsymbol{v}, \lambda_h(t)) = \langle \mathcal{G}_h(t), \boldsymbol{v}\rangle \qquad \forall \boldsymbol{v} \in X_h(\Omega).$$

By differentiating the last equation we obtain that $\lambda_h(t)$ solves (5.1).

Finally, since $\mathbf{grad}(M_h(\Omega_{\rm d})) \subset X_{0,h}(\Omega_{\rm d})$, we are allowed to test the first equation of (5.1) with $\mathbf{grad}\,\lambda_h(t)$ extended by zero to the whole Ω to obtain

$$\frac{d}{dt} b(\widetilde{\mathbf{grad}}\,\lambda_h(t), \lambda_h(t)) = (\boldsymbol{f}(t), \mathbf{grad}\,\lambda_h(t))_{0,\Omega_{\rm d}} = 0.$$

Therefore,

$$\varepsilon_0 |\lambda_h(t)|^2_{1,\Omega_{\rm d}} = b(\widetilde{\mathbf{grad}}\,\lambda_h(t), \lambda_h(t)) = \langle \mathcal{G}_h(0), \widetilde{\mathbf{grad}}\,\lambda_h(0)\rangle = 0$$

and the result follows. □

5.1. **Error estimates.** Our next goal is to prove error estimates for our semidiscrete scheme. Notice that as $\lambda = \lambda_h = 0$, we will only be concerned with error estimates for the main variable $\boldsymbol{u}$.

Consider the linear projection operator $\Pi_h : \mathbf{H}_0(\mathbf{curl}, \Omega) \to V_{0,h}(\Omega)$ defined by

$$\Pi_h \boldsymbol{v} \in V_{0,h}(\Omega): \qquad (\Pi_h \boldsymbol{v}, \boldsymbol{z})_{\mathbf{H}(\mathbf{curl},\Omega)} = (\boldsymbol{v}, \boldsymbol{z})_{\mathbf{H}(\mathbf{curl},\Omega)} \qquad \forall \boldsymbol{z} \in V_{0,h}(\Omega).$$

Lemma 5.6. *There exists $C > 0$, independent of h, such that*

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C \inf_{\boldsymbol{z}\in X_h(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \tag{5.6}$$

for all $\boldsymbol{v} \in V_0(\Omega)$.

Proof. From the definition of Π_h we deduce that

$$\|\boldsymbol{v} - \Pi_h \boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq \inf_{\boldsymbol{z}\in V_{0,h}(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega)}.$$

Since b satisfies the continuous inf-sup condition (cf. the proof of Theorem 4.4) and $\boldsymbol{v} \in V_0(\Omega)$, we can use the trick given in [13, Theorem II-1.1] to conclude that the right-hand side of the previous inequality satisfies

$$\inf_{\boldsymbol{z}\in V_{0,h}(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C \inf_{\boldsymbol{z}\in X_h(\Omega)} \|\boldsymbol{v} - \boldsymbol{z}\|_{\mathbf{H}(\mathbf{curl},\Omega)},$$

which proves (5.6). □

In order to obtain the error estimates, from now on we assume that for almost every $\boldsymbol{x} \in \Omega$, $\mu(\boldsymbol{x}, t)$ is differentiable with respect to t and that there exists a constant $\widetilde{\mu}_1 > 0$ such that

$$|\partial_t \mu(\boldsymbol{x}, t)| \leq \widetilde{\mu}_1 \qquad \forall t \in (0, T), \quad \text{a.e. } \boldsymbol{x} \in \Omega.$$

Lemma 5.7. *Let $\boldsymbol{\rho}_h(t) := \boldsymbol{u}(t) - \Pi_h \boldsymbol{u}(t)$ and $\boldsymbol{\delta}_h(t) := \Pi_h \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$. There exists a constant $C > 0$, independent of h, such that*

$$\begin{aligned} &\sup_{t\in[0,T]} \|\boldsymbol{\delta}_h(t)\|^2_\sigma + \sup_{t\in[0,T]} \|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{0,\Omega} \\ &\quad + \int_0^T \|\,\mathbf{curl}\,\boldsymbol{\delta}_h(t)\|^2_{0,\Omega}\,dt + \int_0^T \|\partial_t \boldsymbol{\delta}_h(t)\|^2_\sigma\,dt \\ &\qquad \leq C \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}\,dt + \sup_{t\in(0,T)} \|\boldsymbol{\rho}_h(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \right\}. \end{aligned} \tag{5.7}$$

Proof. A straightforward computation yields

$$\begin{aligned}(5.8)\qquad &(\partial_t \boldsymbol{\delta}_h(t), \boldsymbol{v})_\sigma + a(t; \boldsymbol{\delta}_h(t), \boldsymbol{v}) \\ &\quad = -(\partial_t \boldsymbol{\rho}_h(t), \boldsymbol{v})_\sigma - a(t; \boldsymbol{\rho}_h(t), \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in V_{0,h}(\Omega).\end{aligned}$$

By taking $\boldsymbol{v} = \boldsymbol{\delta}_h(t)$ in the last identity and using the Cauchy-Schwarz inequality together with (3.8), we obtain

$$\frac{d}{dt}\|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \mu_1^{-1}\|\operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \leq \|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \|\partial_t \boldsymbol{\rho}_h(t)\|_\sigma^2 + \frac{\mu_1}{\mu_0^2}\|\operatorname{\mathbf{curl}} \boldsymbol{\rho}_h(t)\|_{0,\Omega}^2.$$

We now integrate over $[0, t]$ (note that $\boldsymbol{\delta}_h(0) = \mathbf{0}$) and use Gronwall's inequality to obtain

$$\begin{aligned}(5.9)\qquad &\|\boldsymbol{\delta}_h(t)\|_\sigma^2 + \mu_1^{-1}\int_0^t \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(s)\|_{0,\Omega}^2\, ds \\ &\quad \leq C_1 \int_0^T \left[\|\partial_t \boldsymbol{\rho}_h(s)\|_\sigma^2 + \|\boldsymbol{\rho}_h(s)\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\right] ds.\end{aligned}$$

Analogously, taking $\boldsymbol{v} = \partial_t \boldsymbol{\delta}_h(t)$ in (5.8) and using the identity

$$a(t; \boldsymbol{z}, \partial_t \boldsymbol{w}) = \frac{d}{dt} a(t; \boldsymbol{z}, \boldsymbol{w}) - a(t; \partial_t \boldsymbol{z}, \boldsymbol{w}) + \int_\Omega \frac{\partial_t \mu(t)}{\mu(t)^2} \operatorname{\mathbf{curl}} \boldsymbol{z} \cdot \operatorname{\mathbf{curl}} \boldsymbol{w},$$

we obtain

$$\begin{aligned}&\|\partial_t \boldsymbol{\delta}_h(t)\|_\sigma^2 + \frac{1}{2}\frac{d}{dt} a(t; \boldsymbol{\delta}_h(t), \boldsymbol{\delta}_h(t)) + \frac{1}{2}\int_\Omega \frac{\partial_t \mu(t)}{\mu(t)^2} |\operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(t)|^2 \\ &\quad = -(\partial_t \boldsymbol{\rho}_h(t), \partial_t \boldsymbol{\delta}_h(t))_\sigma - \frac{d}{dt} a(t; \boldsymbol{\rho}_h(t), \boldsymbol{\delta}_h(t)) + \int_\Omega \frac{1}{\mu(t)} \operatorname{\mathbf{curl}} \partial_t \boldsymbol{\rho}_h(t) \cdot \operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(t) \\ &\qquad - \int_\Omega \frac{\partial_t \mu(t)}{\mu(t)^2} \operatorname{\mathbf{curl}} \boldsymbol{\rho}_h(t) \cdot \operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(t).\end{aligned}$$

Integrating over $[0, t]$ and using the Cauchy-Schwarz inequality lead to

$$\begin{aligned}&\int_0^t \|\partial_t \boldsymbol{\delta}_h(s)\|_\sigma^2\, ds + \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \\ &\quad \leq C_2 \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(s)\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\, ds + \sup_{s\in[0,T]} \|\operatorname{\mathbf{curl}} \boldsymbol{\rho}_h(s)\|_{0,\Omega}^2 \right. \\ &\qquad \left. + \int_0^t \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(s)\|_{0,\Omega}^2\, ds \right\}.\end{aligned}$$

Finally, using Gronwall's lemma, we have

$$\begin{aligned}&\int_0^t \|\partial_t \boldsymbol{\delta}_h(s)\|_\sigma^2\, ds + \|\operatorname{\mathbf{curl}} \boldsymbol{\delta}_h(t)\|_{0,\Omega}^2 \\ &\quad \leq C_3 \left\{ \int_0^T \|\partial_t \boldsymbol{\rho}_h(s)\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\, ds + \sup_{s\in[0,T]} \|\operatorname{\mathbf{curl}} \boldsymbol{\rho}_h(s)\|_{0,\Omega}^2 \right\}.\end{aligned}$$

The last inequality and (5.9) yield (5.7). □

Theorem 5.8. *Assume that* $\boldsymbol{u} \in \mathrm{H}^1(0,T;\mathbf{H}(\mathbf{curl},\Omega))$ *and let* $\boldsymbol{e}_h(t) := \boldsymbol{u}(t) - \boldsymbol{u}_h(t)$. *There exists* $C>0$*, independent of* h*, such that*

$$\sup_{t\in[0,T]} \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \int_0^T \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}\,dt + \int_0^T \|\partial_t \boldsymbol{e}_h(t)\|^2_\sigma\,dt$$
$$\leq C \left\{ \int_0^T \left[\inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t)-\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t \boldsymbol{u}(t)-\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \right] dt \right.$$
$$\left. + \sup_{t\in[0,T]} \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t)-\boldsymbol{v}\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} \right\}.$$

Proof. Notice that the regularity assumption on $\boldsymbol{u}$ allows us to commute the time derivative and Π_h:

$$\partial_t \left(\Pi_h \boldsymbol{u}(t)\right) = \Pi_h(\partial_t \boldsymbol{u}(t)).$$

Hence, Lemma 5.6 implies that

$$\|\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\boldsymbol{u}(t)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \tag{5.10}$$

and

$$\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C \inf_{\boldsymbol{v}\in X_h(\Omega)} \|\partial_t\boldsymbol{u}(t)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}. \tag{5.11}$$

Thus, the result follows by writing $\boldsymbol{e}_h(t) = \boldsymbol{\rho}_h(t) + \boldsymbol{\delta}_h(t)$ and using the estimates for $\boldsymbol{\delta}_h(t)$ from Lemma 5.7. □

For any $r \geq 0$, we consider the Sobolev space

$$\mathbf{H}^r(\mathbf{curl}, Q) := \left\{\boldsymbol{v} \in \mathrm{H}^r(Q)^3 :\ \mathbf{curl}\,\boldsymbol{v} \in \mathrm{H}^r(Q)^3\right\},$$

endowed with the norm $\|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl},Q)} := \|\boldsymbol{v}\|^2_{r,Q} + \|\mathbf{curl}\,\boldsymbol{v}\|^2_{r,Q}$, where Q is either $\Omega_{\rm c}$ or $\Omega_{\rm d}$. It is well known that the Nédélec interpolant $\mathcal{I}_h\boldsymbol{v} \in X_h(Q)$ is well defined for any $\boldsymbol{v} \in \mathbf{H}^r(\mathbf{curl}, Q)$ with $r > 1/2$; see for instance [2, Lemma 5.1] or [4, Lemma 4.7]. We now fix an index $r > 1/2$ and introduce the space

$$\boldsymbol{\mathcal{X}} := \{\boldsymbol{v} \in \mathbf{H}(\mathbf{curl},\Omega) :\quad \boldsymbol{v}|_{\Omega_{\rm c}} \in \mathbf{H}^r(\mathbf{curl},\Omega_{\rm c}) \text{ and } \boldsymbol{v}|_{\Omega_{\rm d}} \in \mathbf{H}^r(\mathbf{curl},\Omega_{\rm d})\} \tag{5.12}$$

endowed with the broken norm

$$\|\boldsymbol{v}\|_{\boldsymbol{\mathcal{X}}} := (\|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl},\Omega_{\rm c})} + \|\boldsymbol{v}\|^2_{\mathbf{H}^r(\mathbf{curl},\Omega_{\rm d})})^{1/2}.$$

Then, the Nédélec interpolation operator $\mathcal{I}_h : \boldsymbol{\mathcal{X}} \to X_h(\Omega)$ is uniformly bounded and the following interpolation error estimate holds true (see [6, Lemma 5.1] or [2, Proposition 5.6]):

$$\|\boldsymbol{v} - \mathcal{I}_h\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)} \leq C h^{\min\{r,m\}} \|\boldsymbol{v}\|_{\boldsymbol{\mathcal{X}}} \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{X}}. \tag{5.13}$$

Corollary 5.9. *If* $\boldsymbol{u} \in \mathrm{H}^1(0,T;\boldsymbol{\mathcal{X}} \cap \mathbf{H}_0(\mathbf{curl},\Omega))$*, then*

$$\sup_{t\in[0,T]} \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \int_0^T \|\boldsymbol{e}_h(t)\|^2_{\mathbf{H}(\mathbf{curl},\Omega)}\,dt + \int_0^T \|\partial_t \boldsymbol{e}_h(t)\|^2_\sigma\,dt$$
$$\leq C h^{2l} \left\{ \sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}} + \int_0^T \|\partial_t \boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}}\,dt \right\}$$

with $l := \min\{r, m\}$.

Proof. This is a direct consequence of Theorem 5.8 and the interpolation error estimate (5.13). ☐

Remark 5.10. The eddy currents $\sigma\boldsymbol{E}(\boldsymbol{x},t) = \sigma\partial_t\boldsymbol{u}(\boldsymbol{x},t)$ can be approximated by $\sigma\boldsymbol{E}_h(\boldsymbol{x},t)$, where $\boldsymbol{E}_h(\boldsymbol{x},t) := \partial_t\boldsymbol{u}_h(\boldsymbol{x},t)$. In fact, Theorem 5.8 and Corollary 5.9 provide convergence estimates for $\sigma\boldsymbol{E} - \sigma\boldsymbol{E}_h$ in the $\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega_\mathrm{c}))$-norm. On the other hand, by virtue of (3.11), Theorem 5.8 and Corollary 5.9 also yield estimates for the approximation of the magnetic induction $\boldsymbol{B} := \mu\boldsymbol{H}$.

6. Analysis of a fully-discrete scheme

We consider a uniform partition $\{t_n := n\Delta t : \ n = 0,\dots,N\}$ of $[0,T]$ with a step size $\Delta t := \frac{T}{N}$. For any finite sequence $\{\theta^n : \ n = 0,\dots,N\}$, let

$$\bar\partial\theta^n := \frac{\theta^n - \theta^{n-1}}{\Delta t}, \qquad n = 1,2,\dots,N.$$

The fully-discrete version of problem (3.18) reads as follows:
Find $(\boldsymbol{u}_h^n,\lambda_h^n) \in X_h(\Omega)\times M_h(\Omega_\mathrm{d})$, $n = 1,\dots,N$, such that

$$\begin{aligned}(\bar\partial\boldsymbol{u}_h^n, v)_\sigma + b(v,\bar\partial\lambda_h^n) + a(t_n;\boldsymbol{u}_h^n,v) &= (\boldsymbol{f}(t_n),v)_{0,\Omega} \qquad \forall v \in X_h(\Omega),\\ b(\boldsymbol{u}_h^n,\mu) &= 0 \qquad \forall\mu \in M_h(\Omega_\mathrm{d}),\\ \boldsymbol{u}_h^0|_{\Omega_\mathrm{c}} &= \boldsymbol{0},\\ \lambda_h^0 &= 0.\end{aligned} \tag{6.1}$$

Hence, at each iteration step we have to find $(\boldsymbol{u}_h^n,\lambda_h^n) \in X_h(\Omega)\times M_h(\Omega_\mathrm{d})$ such that

$$\begin{aligned}(\boldsymbol{u}_h^n,\boldsymbol{v})_\sigma + \Delta t\, a(t_n;\boldsymbol{u}_h^n,\boldsymbol{v}) + b(\boldsymbol{v},\lambda_h^n) &= F_n(\boldsymbol{v}) \qquad \forall\boldsymbol{v} \in X_h(\Omega),\\ b(\boldsymbol{u}_h^n,\mu) &= 0 \qquad \forall\mu \in M_h(\Omega_\mathrm{d}),\end{aligned}$$

where

$$F_n(\boldsymbol{v}) := \Delta t(\boldsymbol{f}(t_n),\boldsymbol{v})_{0,\Omega} + (\boldsymbol{u}_h^{n-1},\boldsymbol{v})_\sigma + b(\boldsymbol{v},\lambda_h^{n-1}).$$

The existence and uniqueness of $(\boldsymbol{u}_h^n,\lambda_h^n)$ is a direct consequence of the Babuška-Brezzi theory. Indeed, as shown in the proof of Theorem 5.5, the bilinear form b satisfies the discrete inf-sup condition and $\mathcal{A}(\boldsymbol{v},\boldsymbol{w}) := (\boldsymbol{v},\boldsymbol{w})_\sigma + \Delta t\, a(t_n;\boldsymbol{v},\boldsymbol{w})$ induces a norm on its kernel $V_{0,h}(\Omega)$ (cf. Lemma 5.4). Furthermore, testing the first equation of (6.1) with $\widetilde{\mathbf{grad}}\,\lambda_h^n$ and taking into account (4.13) leads to

$$\varepsilon_0|\lambda_h^n|^2_{1,\Omega_\mathrm{d}} = b(\mathbf{grad}\,\lambda_h^n,\lambda_h^n) = b(\mathbf{grad}\,\lambda_h^n,\lambda_h^{n-1}), \qquad n = 1,\dots,N.$$

Consequently, the condition $\lambda_h^0 = 0$ implies that

$$\lambda_h^n = 0, \qquad n = 1,\dots,N.$$

6.1. Error estimates.

Lemma 6.1. *Let $\boldsymbol{\rho}^n := \boldsymbol{u}(t_n) - \Pi_h\boldsymbol{u}(t_n)$, $\boldsymbol{\delta}^n := \Pi_h\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n$ and $\boldsymbol{\tau}^n := \bar\partial\boldsymbol{u}(t_n) - \partial_t\boldsymbol{u}(t_n)$. There exists a constant $C > 0$, independent of h and Δt, such that*

(6.2)

$$\begin{aligned}&\|\boldsymbol{\delta}^n\|^2_\sigma + \|\,\mathbf{curl}\,\boldsymbol{\delta}^n\|^2_{0,\Omega} + \Delta t\sum_{k=1}^n \|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|^2_{0,\Omega} + \Delta t\sum_{k=1}^n \|\bar\partial\boldsymbol{\delta}^k\|^2_\sigma\\ &\qquad\le C\Delta t\left(\sum_{k=1}^N \|\bar\partial\boldsymbol{\rho}^k\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \sum_{k=1}^N \|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|^2_{0,\Omega} + \sum_{k=1}^N \|\boldsymbol{\tau}^k\|^2_\sigma\right),\end{aligned}$$

for all $n = 1,\dots,N$.

Proof. It is straightforward to show that

$$(\bar{\partial}\boldsymbol{\delta}^k,\boldsymbol{v})_\sigma + a(t_k;\boldsymbol{\delta}^k,\boldsymbol{v}) = -(\bar{\partial}\boldsymbol{\rho}^k,\boldsymbol{v})_\sigma - a(t_k;\boldsymbol{\rho}^k,\boldsymbol{v}) + (\boldsymbol{\tau}^k,\boldsymbol{v})_\sigma \qquad \forall \boldsymbol{v}\in V_{0,h}. \tag{6.3}$$

Choosing $\boldsymbol{v}=\boldsymbol{\delta}^k$ in the last identity and using the estimates

$$a(t_k;\boldsymbol{\delta}^k,\boldsymbol{\delta}^k) \geq \mu_1^{-1}\|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega}^2 \quad \text{and} \quad (\bar{\partial}\boldsymbol{\delta}^k,\boldsymbol{\delta}^k)_\sigma \geq \frac{1}{2\Delta t}\left(\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2\right),$$

together with the Cauchy-Schwarz inequality, yield

$$\begin{aligned}&\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 + \Delta t\,\mu_1^{-1}\|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega}^2 \\ &\qquad \leq \frac{\Delta t}{2T}\|\boldsymbol{\delta}^k\|_\sigma^2 + C_1\Delta t\left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega} + \|\boldsymbol{\tau}^k\|_\sigma^2\right).\end{aligned} \tag{6.4}$$

In particular,

$$\|\boldsymbol{\delta}^k\|_\sigma^2 - \|\boldsymbol{\delta}^{k-1}\|_\sigma^2 \leq \frac{\Delta t}{2T}\|\boldsymbol{\delta}^k\|_\sigma^2 + C_1\Delta t\left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega} + \|\boldsymbol{\tau}^k\|_\sigma^2\right).$$

Then, summing over k and using the discrete Gronwall's lemma (see, for instance, [21, Lemma 1.4.2]) lead to

$$\|\boldsymbol{\delta}^n\|_\sigma^2 \leq C_2\Delta t\sum_{k=1}^n\left(\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega}^2 + \|\boldsymbol{\tau}^k\|_\sigma^2\right),$$

for $n=1,\dots,N$. Inserting the last inequality in (6.4) and summing over k we have the estimate

$$\begin{aligned}&\|\boldsymbol{\delta}^n\|_\sigma^2 + \Delta t\sum_{k=1}^n\|\,\mathbf{curl}\,\boldsymbol{\delta}^k\|_{0,\Omega}^2 \\ &\qquad \leq C_3\Delta t\left(\sum_{k=1}^n\|\bar{\partial}\boldsymbol{\rho}^k\|_\sigma^2 + \sum_{k=1}^n\|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega}^2 + \sum_{k=1}^n\|\boldsymbol{\tau}^k\|_\sigma^2\right).\end{aligned} \tag{6.5}$$

Let us now take $\boldsymbol{v}=\bar{\partial}\boldsymbol{\delta}^k$ in (6.3):

$$\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 + a(t_k;\boldsymbol{\delta}^k,\bar{\partial}\boldsymbol{\delta}^k) = -(\bar{\partial}\boldsymbol{\rho}^k,\bar{\partial}\boldsymbol{\delta}^k)_\sigma - a(t_k;\boldsymbol{\rho}^k,\bar{\partial}\boldsymbol{\delta}^k) + (\boldsymbol{\tau}^k,\bar{\partial}\boldsymbol{\delta}^k)_\sigma. \tag{6.6}$$

Since the bilinear form $a(t_k;\cdot,\cdot)$ is nonnegative, we have that

$$\begin{aligned}a(t_k;\boldsymbol{\delta}^k,\bar{\partial}\boldsymbol{\delta}^k) &\geq \frac{1}{2\Delta t}\left[a(t_k;\boldsymbol{\delta}^k,\boldsymbol{\delta}^k) - a(t_k;\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1})\right] \\ &= \frac{1}{2\Delta t}\left[a(t_k;\boldsymbol{\delta}^k,\boldsymbol{\delta}^k) - a(t_{k-1};\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1})\right] \\ &\quad + \frac{1}{2\Delta t}\left[a(t_{k-1};\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1}) - a(t_k;\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1})\right].\end{aligned}$$

Then, there exists $\xi_k\in(t_{k-1},t_k)$ such that

$$\begin{aligned}a(t_k;\boldsymbol{\delta}^k,\bar{\partial}\boldsymbol{\delta}^k) &\geq \frac{1}{2\Delta t}\left[a(t_k;\boldsymbol{\delta}^k,\boldsymbol{\delta}^k) - a(t_{k-1};\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1})\right] \\ &\quad + \frac{1}{2}\int_\Omega\frac{\mu'(\xi_k)}{\mu(\xi_k)^2}|\,\mathbf{curl}\,\boldsymbol{\delta}^{k-1}|^2.\end{aligned} \tag{6.7}$$

On the other hand, a straightforward computation shows that

$$\begin{aligned}a(t_k;\boldsymbol{\rho}^k,\bar{\partial}\boldsymbol{\delta}^k) &= \frac{1}{\Delta t}\left[a(t_k;\boldsymbol{\rho}^k,\boldsymbol{\delta}^k) - a(t_{k-1};\boldsymbol{\rho}^{k-1},\boldsymbol{\delta}^{k-1})\right] - a(t_k;\bar{\partial}\boldsymbol{\rho}^k,\boldsymbol{\delta}^{k-1}) \\ &\quad + \frac{1}{2}\int_\Omega\frac{\mu'(\xi_k)}{\mu(\xi_k)^2}\,\mathbf{curl}\,\boldsymbol{\rho}^{k-1}\cdot\mathbf{curl}\,\boldsymbol{\delta}^{k-1}.\end{aligned} \tag{6.8}$$

Hence, using (6.7) and (6.8) in (6.6), the Cauchy-Schwarz inequality leads to

$$\begin{aligned}\Delta t\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 &+ a(t_k;\boldsymbol{\delta}^k,\boldsymbol{\delta}^k) - a(t_{k-1};\boldsymbol{\delta}^{k-1},\boldsymbol{\delta}^{k-1})\\ &\leq C_4\Delta t\left[\|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + \|\boldsymbol{\tau}^k\|_\sigma^2 + \|\,\mathbf{curl}\,\boldsymbol{\rho}^{k-1}\|_{0,\Omega}^2 + \|\,\mathbf{curl}\,\boldsymbol{\delta}^{k-1}\|_{0,\Omega}^2\right]\\ &\quad - \left[a(t_k;\boldsymbol{\rho}^k,\boldsymbol{\delta}^k) - a(t_{k-1};\boldsymbol{\rho}^{k-1},\boldsymbol{\delta}^{k-1})\right].\end{aligned}$$

Summing over k and using the Cauchy-Schwarz inequality and (6.5), we have

$$\begin{aligned}\Delta t\sum_{k=1}^{n}\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2 &+ \frac{1}{2\mu_1}\|\,\mathbf{curl}\,\boldsymbol{\delta}^n\|_{0,\Omega}^2\\ &\leq C_5\Delta t\left[\sum_{k=1}^{n}\|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + \sum_{k=1}^{n}\|\boldsymbol{\tau}^k\|_\sigma^2 + \sum_{k=1}^{n}\|\,\mathbf{curl}\,\boldsymbol{\rho}^k\|_{0,\Omega}^2\right].\end{aligned}$$

Finally, the result follows by combining the last inequality with (6.5). □

Theorem 6.2. *Assume that $\boldsymbol{u}\in \mathrm{H}^2(0,T;\mathbf{H}(\mathbf{curl},\Omega))$ and let $\boldsymbol{e}^n := \boldsymbol{u}(t_n)-\boldsymbol{u}_h^n$. Then, there exists a constant $C>0$, independent of h and Δt, such that*

$$\begin{aligned}\max_{1\leq n\leq N}\|\boldsymbol{e}^n\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 &+ \Delta t\sum_{k=1}^{N}\|\boldsymbol{e}^k\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + \Delta t\sum_{k=1}^{N}\|\bar{\partial}\boldsymbol{e}^k\|_\sigma^2\\ \leq C\Bigg\{&\max_{1\leq n\leq N}\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\boldsymbol{u}(t_n)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + \Delta t\sum_{n=1}^{N}\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\boldsymbol{u}(t_n)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\\ &+\int_0^T\left(\inf_{\boldsymbol{v}\in X_h(\Omega)}\|\partial_t\boldsymbol{u}(t)-\boldsymbol{v}\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\right)dt + \Delta t^2\int_0^T\|\partial_{tt}\boldsymbol{u}(t)\|_\sigma^2\,dt\Bigg\}.\end{aligned}$$

Proof. A Taylor expansion shows that

$$\sum_{k=1}^{n}\|\boldsymbol{\tau}^k\|_\sigma^2 = \sum_{k=1}^{n}\left\|\frac{1}{\Delta t}\int_{t_{k-1}}^{t_k}(t_{k-1}-t)\partial_{tt}\boldsymbol{u}(t)\,dt\right\|_\sigma^2 \leq \Delta t\int_0^T\|\partial_{tt}\boldsymbol{u}(t)\|_\sigma^2\,dt. \tag{6.9}$$

Moreover,

$$\begin{aligned}\sum_{k=1}^{n}\|\bar{\partial}\boldsymbol{\rho}^k\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 &\leq \frac{1}{\Delta t}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\,dt\\ &\leq \frac{1}{\Delta t}\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\,dt.\end{aligned} \tag{6.10}$$

Combining (6.2), (6.9), and (6.10) and recalling that $\|\cdot\|_{V_0(\Omega)}$ is equivalent to $\|\cdot\|_{\mathbf{H}(\mathbf{curl},\Omega)}$ in $V_{0,h}(\Omega)$, we obtain

$$\begin{aligned}\max_{1\leq n\leq N}\|\boldsymbol{\delta}^n\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 &+ \Delta t\sum_{k=1}^{N}\|\boldsymbol{\delta}^k\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2 + \Delta t\sum_{k=1}^{N}\|\bar{\partial}\boldsymbol{\delta}^k\|_\sigma^2\\ &\leq C_0\left\{\int_0^T\|\partial_t\boldsymbol{\rho}_h(t)\|_{\mathbf{H}(\mathbf{curl},\Omega)}^2\,dt + \Delta t\sum_{k=1}^{N}\|\,\mathbf{curl}\,\boldsymbol{\rho}_h(t_k)\|_{0,\Omega}^2\right.\\ &\qquad\left. +(\Delta t)^2\int_0^T\|\partial_{tt}\boldsymbol{u}(s)\|_\sigma^2\,ds\right\}.\end{aligned}$$

The result follows from the fact that $\boldsymbol{e}^n = \boldsymbol{\delta}^n + \boldsymbol{\rho}^n$ and the triangle inequality. □

Finally, we deduce from (5.10), (5.11), and (5.13) the following asymptotic error estimate.

Corollary 6.3. *Under the assumptions of Corollary* 5.9 *and Theorem* 6.2*, there exists a constant C, independent of h and Δt, such that*

$$\max_{1\le n\le N} \|\boldsymbol{e}^n\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \Delta t \sum_{k=1}^{N} \|\boldsymbol{e}^k\|^2_{\mathbf{H}(\mathbf{curl},\Omega)} + \Delta t \sum_{k=1}^{N} \|\bar{\partial}\boldsymbol{e}^k\|^2_{\sigma}$$
$$\le C h^{2l} \left\{ \max_{1\le n\le N} \|\boldsymbol{u}(t_n)\|^2_{\boldsymbol{\mathcal{X}}} + \int_0^T \|\partial_t \boldsymbol{u}(t)\|^2_{\boldsymbol{\mathcal{X}}}\, dt \right\}$$
$$+ C(\Delta t)^2 \int_0^T \|\partial_{tt}\boldsymbol{u}(t)\|^2_{\sigma}\, dt,$$

with $l := \min\{m, r\}$.

Remark 6.4. At each time step $t = t_k$, we can approximate the eddy currents $\sigma\boldsymbol{E}(\boldsymbol{x}, t_k)$ by $\sigma\boldsymbol{E}_h^k$, where $\boldsymbol{E}_h^k := \bar{\partial}\boldsymbol{u}_h^k$. In fact, Corollary 6.3 yields the following convergence estimate in a discrete $\mathrm{L}^2(0, T; \mathrm{L}^2(\Omega_{\mathrm{c}}))$-norm:

$$\Delta t \sum_{k=1}^{N} \|\sigma\boldsymbol{E}(t_k) - \sigma\boldsymbol{E}_h^k\|^2_{0,\Omega_{\mathrm{c}}} \le C\left[h^{2l} + (\Delta t)^2\right].$$

7. Conclusions

We have introduced an $\boldsymbol{E}$-based formulation for the time-dependent eddy current problem in a bounded domain. The variables of the formulation are a time-primitive of the electric field and a Lagrange multiplier used to impose the divergence-free constraint in the dielectric domain. We have shown that this formulation is well posed and that the Lagrange multiplier vanishes identically.

Then, we have proposed a finite element space discretization based on Nédélec edge elements for the main variable and standard nodal finite elements for the Lagrange multiplier. We have proved the well-posedness of the resulting semi-discrete scheme as well as optimal order error estimates. The discrete Lagrange multiplier has been proved to vanish, as well. Finally we have analyzed an implicit time-discretization scheme. Under appropriate smoothness assumptions, we have proved that the fully discrete problem also converges with optimal order. This approach provides suitable approximations of the quantities of typical interest: the eddy currents in the electric domain and the magnetic induction.

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